Tutorial Math 2040A 7th Week

§5.2 Diagonalizability

7. For $A = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix} \in M_{2\times 2}(R)$, find an expression for A^n where n is an arbitrary positive integer.

Solution: Diagonalize A: $Q^{-1} A Q = D$ $\quad Q = \begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix}$ $D = \begin{pmatrix} 5 & 0 \\ 0 & -1 \end{pmatrix}$

$$A^n = Q D^n Q^{-1} = Q \begin{pmatrix} 5^n & 0 \\ 0 & (-1)^n \end{pmatrix} Q^{-1}$$

9. T - a linear operator on a finite-dimensional v.s. V. Suppose $\exists$ an ordered basis β for V s.t. $[T]_\beta$ is an upper triagular matrix.

a) Prove that the characteristic polynomial for T splits.

b) State and prove an analogous result for matrices.

Solution: Characteristic polyomial of T is independent of the choice of β (Why? – We have already proved this last time). $\Rightarrow$

$$f(t) = \det([T]_\beta - tI) = \prod_{i=1}^{n} (([T]_\beta)_{ii} - t) \text{ splits.}$$

(Why? – Because it is ~~an~~ upper triangular)

11. Let A be an $n \times n$ matrix that is similar to an upper triangular matrix and has the distinct eigenvalues $\lambda_1 \cdots \lambda_k$ with corresponding multiplicities $m_1 \cdots m_k$.

Prove that:

a) $\operatorname{tr} A = \sum_{i=1}^{k} m_i \lambda_i$

b) $\det A = (\lambda_1)^{m_1} (\lambda_2)^{m_2} \cdots (\lambda_k)^{m_k}$

Solution:

Important Fact: (Why?)

$\operatorname{tr} A =$ the coefficient of t^{n-1} in the char. polynomial of A

$\det A =$ the constant term in the char. polyn. of A.

18. a) Prove that if T and U are simultaneously diagonalizable operators, then T and U commute.

b) Prove that if A and B are simultaneously diagonalizable matrices, then A and B commute.

Solution: a) Let β be the basis makes T and U simultaneously diagonalizable. Each pair of diagonal matrices commutes.

$[T]_\beta [U]_\beta = [U]_\beta [T]_\beta$ i.e. T and U commutes

b) Q s.t. $Q^{-1}AQ$, $Q^{-1}BQ$ are diagonal.

$(Q^{-1}AQ)(Q^{-1}BQ) = (Q^{-1}BQ)(Q^{-1}AQ)$ i.e.

$AB = BA$

22. Let T be a linear operator on a fin. dim. v.s. V, and Suppose that the distinct eigenvalues of T are $\lambda_1 \cdots \lambda_k$. Prove that

$$\text{Span}(\{x \in V : x \text{ is an eigenvector of } T\}) = E_{\lambda_1} \oplus \cdots \oplus E_{\lambda_k}$$

Solution: Recall the definition of a direct sum of fin. dim v.s. We have to check 2 conditions.

- LHS $= \sum E_{\lambda_i}$
- Let $W = \sum_{i=2}^{k} E_{\lambda_i}$. If $\exists\, v_1 \neq 0$, $v_1 \in E_{\lambda_1} \cap W$, then

① $v_1 + c_2 v_2 + \cdots + c_k v_k = 0$ $\quad c_i$ scalars, $v_i \in E_{\lambda_i}$

Apply T to both sides:

② $0 = T(0) = \lambda_1 v_1 + c_2 \lambda_2 v_2 + \cdots + c_k \lambda_k v_k = 0$

② $-$ ① $\Rightarrow$ $c_2(\lambda_2 - \lambda_1) v_2 + \cdots + c_k(\lambda_k - \lambda_1) v_k = 0$

This is impossible since $\lambda_i - \lambda_1$ is nonzero for all i and c_i cannot be all zero.

Similar for E_{λ_i} $i = 2 \cdots n$

(To be more precise, we use induction on number of components of W i.e. number of E_{λ_i}'s in W. $W = E_{\lambda_1} + E_{\lambda_2}$ case is solved as above. In general the step of "② $-$ ① $\Rightarrow$" uses the induction hypothesis)

§5.4 Invariant Subspaces

2. T linear operator on v.s. V. Determine whether W is a T-invariant subspace of V.

a) $V = P_3(R) \quad T(f(x)) = f'(x). \quad W = P_2(R)$

Yes $\quad T(ax^2+bx+c) = 2ax+b \in W$ for every $ax^2+bx+c \in W$

b) $V = \mathbb{R}^3. \quad T(a,b,c) = (a+b+c, a+b+c, a+b+c).$

$W = \{(t,t,t) : t \in \mathbb{R}\}$

Yes. For every $(t,t,t) \in W$. $T(t,t,t) = (3t,3t,3t) \in W$

e) $V = M_{2\times2}(\mathbb{R}). \quad T(A) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} A \quad W = \{A \in V,\ A^t = A\}$

No. For $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \in W$, $\quad T(A) = \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix} \notin W$

3. T lin. op. on v.s. V. Prove that the following subspace are T-invariant.

a) $\{0\}$ and V

b) $N(T)$, $R(T)$

c) E_λ any eigenvalues λ of T.

Solution: a) $T(0) = 0, \quad T(v) \in V$ for any $v \in V$.

b) $v \in N(T) \Rightarrow T(v) = 0 \in N(T)$

(Why?)

$v \in R(T) \Rightarrow T(v) \in R(T)$ (By def'n)

c) $v \in E_\lambda \Rightarrow T(v) = \lambda v \in E_\lambda$

why? $\quad T(\lambda v) = \lambda(T(v)) = \lambda(\lambda v)$

$\Rightarrow \lambda v \in E_\lambda$

6. T lin. op. on v.s. V. find an ordered basis for the T-cyclic subspace generated by z.

a) $V = \mathbb{R}^4$. $T(a, b, c, d) = (a+b, b-c, a+c, a+d)$ $z = e_1$

Solution: $z = (1, 0, 0, 0)$ $T(z) = (1, 0, 1, 1)$ $T^2(z) = (1, -1, 2, 2)$

$T^3(z) = (0, -3, 3, 3)$.

$\Rightarrow$ dim $= 3$ $\{z, T(z), T^2(z)\}$ is a basis

13. T - lin. op. on v.s. V. v - nonzero vector in V.

W - the T-cyclic subspace of V generated by v.

For any $w \in V$. prove that $w \in W$ iff $\exists$ a polynomial $g(t)$ s.t. $w = g(T)(v)$.

Solution: If $w \in W$. then w is a lin. combination of $\{v, T(v), \cdots\}$ $\Rightarrow$ $w = g(T)(v)$ for some polynomial g.

Conversely, if $w = g(T)(v)$ $\Rightarrow$ w is a lin. combinatio of $\{v, T(v), \cdots\}$ $\Rightarrow$ $w \in W$.